NORTH-HOLLAND

Loop and Cyclic Niche Graphs

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Dedicated to Professor John Maybee on the occasion of 65th birthday.

Submitted by J. Richard Lundgren

ABSTRACT

The niche graph of a digraph is a graph created by taking the vertices of the digraph and connecting any pair which occur in an in-neighborhood or an out-neighborhood of the digraph. Originally the problem of which graphs were the niche graphs of acyclic digraphs received much attention; this paper considers the effect of relaxing the requirement that the digraph be acyclic. Several new classes of graphs turn out to be niche graphs under this relaxation, but unlike the analogous result for competition graphs, many graphs still are not niche graphs, nor can they be made into niche graphs by adding isolated vertices.

1. INTRODUCTION

If $D(V, A)$ is a digraph, the *niche graph* of D is the graph $G(V, E)$ with xy an edge of G if and only if there is a vertex z in D such that xz and yz are arcs of D or zx and zy are arcs of D. Another way of viewing the niche graph is to think of the edges of the graph as joining every pair of vertices which are in a common in-neighborhood and every pair of vertices which are in a common out-neighborhood of the digraph. Note that every (in- or out-) neighborhood of D forms a clique (not necessarily maximal) in G. The members of an in-neighborhood of any vertex of D are said

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to have a *common prey* (and they are joined by *competition edges*); the members of an out-neighborhood, a *common enemy* (and they are joined by *common-enemy edges*). If there is an arc from vertex v to vertex u in D, then v is called a *predator* and u is called a *prey*.

Niche graphs are a variant of competition graphs where only the vertices of in-neighborhoods are adjacent. (There is a good recent survey of known results and open problems about competition graphs and their variants by Lundgren [10].) Despite the close similarity in definitions, niche graphs and competition graphs have quite different properties.

Roberts [11] defined the *competition number* of a graph to be the minimum number of isolated vertices which must be added to the graph before it is the competition graph of an acyclic digraph, and proved that every graph has a finite competition number. A similar definition for niche number was provided by Cable, Jones, Lundgren, and Seager [6]. The *niche number* of a graph $G, n_0(G)$, is the minimum number of isolated vertices such that the disjoint union of G and those vertices is the niche graph of an acyclic digraph. Unfortunately, some graphs do not have a finite niche number, and alternative definitions (which do not conflict with the standard definition of competition number) have been proposed by Anderson, Jones, Lundgren, and Seager [3]. These definitions allow all graphs to have a finite niche number.

Competition graphs arose in the study of food webs, where arcs in the digraph represented one species feeding on another. Since such digraphs tended to be loopless and acyclic, a graph was called a competition graph only if it was the competition graph of an acyclic digraph. Niche graphs were originally defined and studied similarly; a graph was called a niche graph only if it was the niche graph of some acyclic digraph. Later work has relaxed the requirement that the digraph be acyclic. The question of which graphs are the competition graphs of general digraphs was settled in the early eighties by the following two theorems. (An *edge clique cover* of G is a set of cliques of G which includes every edge of G.)

THEOREM 1 (Dutton and Brigham [7]). *A graph G on n vertices is the competition graph of an arbitrary digraph (loops allowed) if and only if G has an edge clique cover of at most n cliques.*

THEOREM 2 (Roberts and Steif [12]). *A graph G on n vertices is the competition graph of an arbitrary loopless digraph if and only if G is not K_2 and has an edge clique cover of at most n cliques.*

These results show that allowing arbitrary digraphs to generate competition graphs reduces the problem of deciding whether a graph is a competition graph to finding an edge clique cover with cardinality at most the number of vertices in the graph. In this paper we will briefly examine related questions for niche graphs: which graphs are the niche graphs of digraphs with cycles allowed, and which are the niche graphs of arbitrary digraphs with both loops and cycles?

These questions do not have the encompassing answers their cognates in competition graph theory possess. It is easy to show, for example, that many classes of graphs still have infinite niche number. Current results are fragmentary, but we show that unit-interval graphs are niche graphs of arbitrary digraphs with loops (no other cycles are necessary), and that a recent class of graphs with large finite original niche number, discovered by Fishburn and Gehrlein, are all niche graphs of digraphs with cycles allowed.

2. LOOP AND CYCLIC NICHE GRAPHS

We will say that a graph G is a *cyclic niche graph* if G is the niche graph of an arbitrary loopless digraph. We define the *cyclic niche number* of a graph $G, n_c(G)$, to be the least number of isolated vertices such that the disjoint union of G with those vertices is a cyclic niche graph. Furthermore, G is a *loop niche graph* if it is the niche graph of an arbitrary digraph with loops allowed; the *loop niche number* of $G, n_l(G)$, is the least number of isolated vertices such that the disjoint union of G and those isolated vertices is a loop niche graph. Note that $n_l(G) \leq n_c(G)$ for all graphs G. As in the case of the original niche number, if a graph cannot be made into a cyclic or loop niche graph by appending isolated vertices, its cyclic or loop niche number is ∞.

The class of cyclic niche graphs is not as all-inclusive as the class of cyclic competition graphs; we will first look at some graphs whose cyclic niche number is infinity.

THEOREM 3. $n_c(K_{1,3}) = \infty$.

Proof. Consider the central vertex (the one adjacent to each of the others). It is incident with three edges. Since the graph is triangle-free, each edge represents a common-enemy or common-prey pair of vertices in any digraph whose niche graph is $K_{1,3}$ (with, perhaps, some isolated vertices). Thus the central vertex must either prey on at least two other vertices, or be preyed on by at least two other vertices. In either case those other vertices are connected by an edge in the niche graph, which is impossible. So $K_{1,3}$ is not a cyclic niche graph, with or without additional isolated vertices, and hence has infinite cyclic niche number. ■

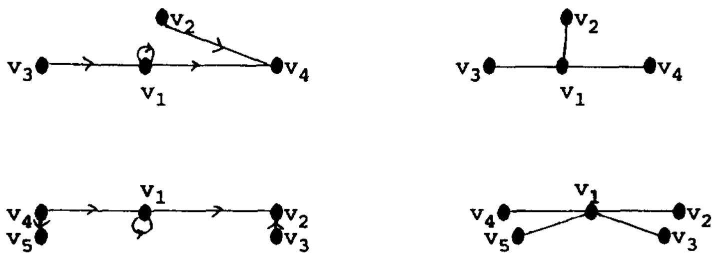

FIG. 1. $K_{1,3}$ and $K_{1,4}$ are loop niche graphs.

The argument above is essentially that of Cable et al. [6] in their original paper, used to prove that $n_0(K_{1,3})$ is infinite. Note that the argument does not apply to loop niche graphs, because the central vertex need not prey on two other vertices; one of the vertices it preys on may be itself. In fact, both $K_{1,3}$ and $K_{1,4}$ are loop niche graphs, as the diagram in Figure 1 illustrates.

Not all stars are loop niche graphs; even $K_{1,5}$ has an infinite loop niche number. This is a result of the following theorem, and the proof is virtually identical to the proof of a similar theorem in Cable et al. [6].

THEOREM 4. *If G is a graph whose largest clique has m vertices, and the maximum degree of G is greater than $2m(m-1)$, then $n_c(G) = n_l(G) = \infty$.*

Proof. Let v be a vertex of G with maximum degree at least $2m(m-1)+1$. Assume G has a finite loop niche number. Since the largest clique of G has m vertices, at most m vertices can prey on (or be preyed on by) any one vertex. If m vertices prey on v, each preys on at most $m-1$ other vertices of G. Then at most $m(m-1)$ vertices can be adjacent to v by common-enemy edges. Similarly, at most $m(m-1)$ vertices can be adjacent to v by competition edges. Thus at most $2m(m-1)$ vertices can be adjacent to v; since the degree of v is greater than that, G cannot be a graph with finite loop niche number. Since every graph with a finite cyclic niche number has a finite loop niche number as well, $n_c(G)$ is also infinite. ∎

This theorem demonstrates that many graphs have infinite cyclic niche numbers; however, some graphs do have cyclic niche number less than their original niche number. The following theorem gives two cases in which graphs are cyclic, but not acyclic, niche graphs. The first result is not surprising. The second result, on cycles, is notable because the cycles C_4, C_5, and C_6 provided the first graphs with (original) niche number 2.

THEOREM 5. *For graphs with three or more vertices,*

(i) *if* $n_0(G) = 1$, *then* $n_c(G) = 0$;
(ii) *if* G *is a cycle, then* $n_c(G) = 0$.

Proof. For part (i), assume that in an acyclic digraph D with its niche graph consisting of G and one isolated vertex, the extra is a prey. (Of course, we can reverse all the arcs in the digraph to make this so.) Then, if all graph vertices prey on the extra (so that G is the complete graph), replace the digraph with the complete symmetric digraph on the vertices of G, and the niche graph of this digraph is G. Otherwise, since D is acyclic, a topological ordering of the vertices of D is guaranteed to exist, in which each vertex preys only on vertices succeeding it. Take the first vertex v which doesn't prey on the extra, and let all the extra's predators prey on v. Note that v had no previous predators, since every vertex preceding it in the tropological ordering preyed on the extra, and therefore had no other prey. Eliminate the extra, and D now has G as its niche graph. The in-neighborhood of v creates the edges previously created by the in-neighborhood of the extra, and since v had no other predators (and the predators of v no other prey), no new edges are created.

To show that the cycle C_n is a cyclic niche graph, create a digraph on the vertices of C_n by having arcs from v_i to v_{i+1} and v_{i+2}, modulo n. ∎

If a graph G has an even (original) niche number n_0, then there is at least one acyclic digraph which has as its niche graph the disjoint union of G and n_0 isolated vertices. If such a digraph has an equal number of predator and prey extras, and no prey of any extra preys on another extra, then we can show that G is a cyclic niche graph. In fact, we may use the following theorem of Bowser and Cable [4] to allow us to prove a stronger statement.

THEOREM 6. *If* $n_0(G) = k$, *and* D *is an acyclic digraph with niche graph* $G \cup I_k$, *then every extra either preys on a graph vertex which does not prey on an extra, or is preyed on by a graph vertex which is not preyed on by an extra.* ∎

THEOREM 7. *If, for a graph* G *with* $n_0(G) = 2k$, *there exists an acyclic digraph* D *with niche graph* $G \cup I_{2k}$ *and the* $2k$ *extra vertices are divided into* k *predators and* k *prey, then* G *is a cyclic niche graph.*

Proof. Assume that D is as described in the theorem. Then match each prey extra with a distinct predator extra. Modify D by replacing each arc from a graph vertex to a prey extra with an arc from the graph vertex to the matched predator extra. Delete the (now isolated) prey extras, and call the resulting digraph D'. Now D' is loopless, since we have only redirected the arcs that went from a graph vertex to an extra so that they now go to a different extra. The niche graph of D' is $G \cup I_k$, since any edge of G that was created by competition for a prey extra is now created by competition for the matched "predator" extra. No new edges have been created, for each vertex preying on an extra had no other prey, and each predator extra had no previous predators. Now by Bowser and Cable's result, among the predators of each extra x_i there is at least one vertex v_i which is not a prey of any extra, and among the prey of x_i is at least one vertex u_i which does not prey on any extra. For each x_i, replace each in-arc vx_i with the arc vu_i, and replace each out-arc x_iu with the arc v_iu. Delete the (now isolated) extras, and call this resulting graph D''. Again, D'' is still loopless, for each new arc vu_i is from a vertex which preyed on an extra in D to a vertex which did not, and each new arc v_iu is from a vertex which did not have an extra as predator to a vertex which did. The niche graph of D'' still contains all the edges of G, since each (in- or out-) neighborhood of an extra is now the (in- or out-) neighborhood of a graph vertex. No new edges have been added, for the graph vertices which received the transferred (in- or out-) neighborhoods had neighborhoods in the original digraph which were single extras, and those have been deleted. Hence the niche graph of D'' is G. ∎

One of the major open problems in the early investigations of niche graphs was whether any graphs had a finite original niche number greater than two. This was answered in the affirmative by Fishburn and Gehrlein [8], who constructed an infinite family of graphs with increasingly large niche numbers. The digraphs they produced to prove that their graphs had finite niche numbers satisfy the conditions of Theorem 7, so this interesting corollary follows:

COROLLARY. *The Fishburn-Gehrlein graphs are cyclic niche graphs, and the difference between $n_0(G)$ and $n_c(G)$ may be arbitrarily large.* ∎

Figure 2 shows one of the Fishburn-Gehrlein graphs, its digraph showing that its original niche number is finite, and the modified digraph showing that the graph is a cyclic niche graph.

By the construction in Theorem 7, all known graphs with even niche number are cyclic niche graphs. However, this does not prove that all

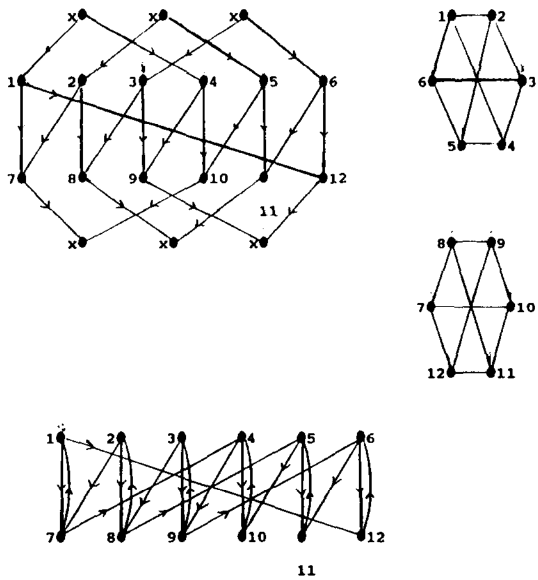

FIG. 2. A Fishburn-Gehrlein graph, its digraph, and its cyclic digraph.

graphs with an even niche number are cyclic niche graphs. If the only digraphs of minimum order which have $G \cup I_k$ as their niche graph are those with an unbalanced number of predator extras and prey extras, we do not know if G is a cyclic niche graph or not. Furthermore, the graphs with odd niche number automatically have an imbalance in the extras of a generating digraph, and the question of whether the smallest known graphs with niche number three (again found by Fishburn and Gehrlein [9]) are cyclic niche graphs is open.

For the final result in this paper, we will consider unit interval graphs. A graph is an *interval graph* if it can be represented as the intersection graph of a family of intervals on the real line (i.e., each vertex is assigned an interval, and two vertices are adjacent if and only if their intervals have a nonempty intersection). A *unit interval graph* is an interval graph such that the intervals can all be of unit length. One useful property of unit interval graphs is that the vertices can be ordered so that each maximal clique consists of consecutively numbered vertices. Anderson, Bowser, Cable, and Lundgren [2] have shown that many subclasses of unit interval graphs are niche graphs, but the conjecture that every noncomplete unit interval graph is a niche graph remains unproved. For loop niche graphs, however, the conjecture is true.

THEOREM 8. *If G is a unit interval graph, $n_l(G) = 0$.*

Proof. Using the usual ordering of the vertices of G, so that each maximal clique of G consists of consecutively numbered vertices, we construct a digraph on $V(G)$. For each maximal clique of G, draw an arc from each vertex of the clique to the highest-numbered vertex of the clique. (There is a loop from the last vertex to itself in each clique.) We claim that the niche graph of this digraph is the unit interval graph G. Each edge of G is in at least one maximal clique and so is in the niche graph as a competition edge, since the vertices of the clique all have a common prey. There are no other competition edges, for the only vertices that are prey are the last in each of the maximal cliques. Suppose there is a common-enemy edge in the niche graph, that is, there is a vertex x that preys on two separate vertices y and z in the digraph, and therefore yz is an edge in G. Say, without loss of generality, that z is the higher-numbered of y and z. Since the vertices of each maximal clique are numbered consecutively, and x and z are in the same clique, y must also be in that clique, so y preys on z. But z preys on z also (it is the last vertex in some maximal clique, since x preys on it). Thus the common-enemy edge yz is also a competition edge, and no common-enemy edge is a new edge. Hence the niche graph of D is G, and therefore every unit interval graph is a loop niche graph. ■

3. OPEN QUESTIONS

These topics are quite new, and many questions have not been investigated. Some possible directions for further research include answering the questions:

Are there graphs with infinite niche numbers but finite cyclic niche numbers?

Are there graphs with finite cyclic niche numbers other than zero? (The Fishburn-Gehrlein eight-vertex graphs with niche number three may provide a positive answer.)

Is there a good characterization of graphs which have finite loop niche number?

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